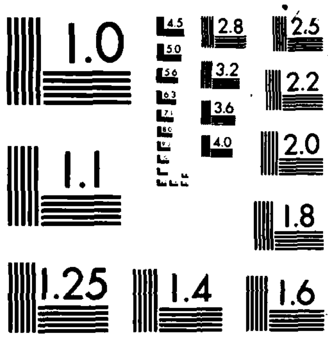
AD-A141 601 L INFINITY STABILITY OF AN EXPONENTIALLY DECREASING SOLUTION OF THE PROBL..(U) WISCONSIN UNIV-MADISON MATHEMATICS RESEARCH CENTER H MATANO APR 84 1/1

UNCLASSIFIED MRC-TSR-2678 DAAG29-80-C-0041 F/G 12/1 NL



MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

2

AD-A141 601

MRC Technical Summary Report #2678

L^{∞} STABILITY OF AN EXPONENTIALLY DECREASING SOLUTION OF THE PROBLEM $\Delta u + f(x,u) = 0$ IN $\mathbf{R}^n$

Hiroshi Matano

Mathematics Research Center
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53705

April 1984

(Received February 27, 1984)

Sponsored by

U. S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

UNIVERSITY OF WISCONSIN-MADISON
MATHEMATICS RESEARCH CENTER

L^∞ STABILITY OF AN EXPONENTIALLY DECREASING SOLUTION OF THE PROBLEM $\Delta u + f(x,u) = 0$ IN $\mathbb{R}^n$

Hiroshi Matano

Technical Summary Report #2678
April 1984

ABSTRACT

We consider the Cauchy problem $u_t = \Delta u + f(x,u)$, $x \in \mathbb{R}^n$, $t > 0$, and prove that if there exist a strict supersolution $\bar{w}$ and a strict subsolution w with $\bar{w} > w$ then there exists at least one stable equilibrium solution between $\bar{w}$ and w provided that f satisfies certain conditions. The stability is with respect to the L^∞ norm. Unlike the case where the spatial domain is bounded, some difficulties occur near $|x| = \infty$ in the present problem. The major part of this paper is devoted to dealing with such difficulties.

AMS (MOS) Subject Classifications: 35K55, 35B35, 35J60

Key Words: Nonlinear parabolic equation, stable solution, super and subsolution, comparison principle

Work Unit Number 1 (Applied Analysis)

Sponsored by the United States Army under Contract No. DAAG29-80-C-0041.

SIGNIFICANCE AND EXPLANATION

The equations we study here arise in many fields of mathematical sciences such as population dynamics in mathematical ecology, population genetics, chemical reaction theory, etc. Our study concerns the stability of equilibrium solutions of these equations.

Among the solutions of nonlinear evolution equations, the practically important ones are those which are stable in a certain sense. However, finding a stable equilibrium solution is in many cases considerably more difficult than just proving the existence of equilibrium solutions. In this paper we give a useful sufficient condition for the existence of stable equilibrium solutions.

The result we present in this paper is a generalization of the author's former results on equations in bounded domains. However, the equations we consider here (which are in the whole space $\mathbb{R}^n$) exhibit much more complicated dynamical behavior, and therefore only a few results have been known about the existence of stable equilibrium solutions. The objective of this paper is to make a systematic study of these equations and to give rather a general theorem on the existence of stable equilibrium solutions.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

L^∞ STABILITY OF AN EXPONENTIALLY DECREASING SOLUTION OF THE PROBLEM $\Delta u + f(x,u) = 0$ IN $\mathbb{R}^n$

Hiroshi Matano

1. Introduction

In the earlier paper [5] the author has studied the dynamical structure of rather a wide class of equations in which a certain stronger version of comparison principle holds. Such equations, characterized as strongly order-preserving local semiflows, include single semilinear parabolic equations in bounded domains, weakly coupled reaction diffusion systems of competition type with two unknowns, those of cooperation type with any number of unknowns, etc. An interesting feature of strongly order-preserving local semiflows having a certain compactness property is that any unstable equilibrium point has non-empty unstable manifold. This property, which is not trivial since linearized instability is not assumed, has far greater implications than it apparently seems (see [5] for details). Hirsch [3] has made an independent study of basically the same class of local semiflows and has obtained other interesting results. Among other things, he has proved that almost all the bounded orbits are quasi-convergent; in other words, their ω-limit sets are contained in the set of equilibria. As a consequence, any periodic orbit is unstable.

Another interesting property of such local semiflows is that if $\bar{w}$ is a time-independent strict supersolution (the definition of which will be given in the next section) and if w is a time-independent strict subsolution with $\bar{w} > w$ then there exists at least one stable equilibrium point (i.e. equilibrium solution) between $\bar{w}$ and w ([5; Theorem 3]). This theorem is a generalization of the author's former result [4; Theorem 4.2] on single semilinear diffusion equations in bounded domains, and is

Sponsored by the United States Army under Contract No. DAAG29-80-C-0041.

exceedingly useful in finding stable equilibrium solutions. One of its applications to reaction-diffusion systems of competition type is found in Matano and Mimura [6].

What all the above results show is the usefulness of the strong comparison principle, which until recently has not been appreciated in the context of evolution equations. While the standard comparison principle is not by itself powerful enough to have profound implications on the dynamical behavior of solutions, its slightly stronger version, on the other hand, can play a significantly powerful role in the qualitative analysis of a certain class of equations.

With all these developments in the theory of strongly order-preserving systems, there are nonetheless some practically important equations that are order-preserving (which means that the comparison theorem holds) but not strongly order-preserving in the sense of [3] or [5]. These include initial value problems of the form

$$(1)\qquad \begin{cases} \dfrac{\partial u}{\partial t} = \Delta u + f(x,u), & x \in \mathbb{R}^n,\ t > 0\ , \\ u(x,0) = u_0(x), & x \in \mathbb{R}^n\ , \end{cases}$$

where $\Delta = \partial^2/\partial x_1^2 + \cdots + \partial^2/\partial x_n^2$, and porus media type equations of the form

$$(2)\qquad \frac{\partial u}{\partial t} = \Delta(u^m) + f(x,u)\ ,$$

where $m > 1$. The unboundedness of $\mathbb{R}^n$ in (1) or the degeneracy of diffusion near $u = 0$ in (2) prevent these equations from being strongly order-preserving (although they are both order-preserving), hence the general theory of [3] or [5] does not apply. In order to study these equations we need careful analysis near $|x| = \infty$ or near the free boundary. The aim of the present paper is to show that some of the results obtained for the strongly order-preserving systems still hold true for the problem (1) under certain circumstances. The problem (2) will be studied in the forthcoming paper [2].

Let us consider the initial value problem (1), where the initial data u_0 is a bounded continuous function defined on $\mathbb{R}^n$. We assume $f(x,0) \equiv 0$, so that $u = 0$ is an equilibrium solution of (1). We may call $u = 0$ the trivial equilibrium solution. Various kinds of sufficient conditions are known for the existence of non-trivial

equilibrium solutions. In many cases, however, the harder part of the analysis is to study the stability of those equilibrium solutions.

In an attempt to extend the above-mentioned result [4; Theorem 4.2] (bounded domain case) to the present equation (1), Crandall, Fife and Peletier [1] proved the following: If f satisfies (A.1)-(A.3) (see next section) and if $\bar{w}$ and w are time-independent strict super- and sub-solutions respectively with $\bar{w} > w$ and $\bar{w}(\infty) = w(\infty) = 0$, there exists at least one stable equilibrium solution between $\bar{w}$ and w.

If, in particular, one can find $\bar{w}(x)$ and $w(x)$ such that $\bar{w} > w > 0$, then their result guarantees the existence of a positive stable equilibrium solution that decays as $|x| \to \infty$. However, the stability they discussed was rather a weak one, namely the stability with respect to perturbations decaying rather rapidly as $|x| \to \infty$. The question therefore still remained open as to whether the equilibrium solution they obtained is L^∞ stable or not; in other words, whether or not it is stable under any bounded small perturbation that does not necessarily decay as $|x| \to \infty$.

The major contribution of the present paper is to give an affirmative answer to the above question. The key point of the discussion is the idea of "strong stability" (see Definition 5), which was first introduced in [5]. Since our problem (1) falls outside the category of strongly order-preserving systems, we need some extra careful analysis near $|x| = \infty$.

Notation and the main theorem will be given in Section 2. The proof of the theorem will be carried out in Section 3. Finally, in the Appendix, we give a counterexample that shows the assumptions (A.1)-(A.3) in the Theorem or Proposition 2.3 cannot be dropped; this illustrates the difference between the present problem (1) and an initial-boundary value problem in a bounded domain (the latter being strongly order-preserving).

The author expresses his gratitude to Professor Paul C. Fife for many stimulating discussions.

2. Notation and Main Theorem

We assume the following:

(A.1) $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is a C^2 map, and for any $K > 0$ the derivatives f, f_x, f_u, f_{xu}, f_{uu} are bounded in the region $x \in \mathbb{R}^n$, $|u| \leq K$.

(A.2) $f(x,0) = 0$ for all $x \in \mathbb{R}^n$.

(A.3) There exist positive numbers α, M, δ such that

$$f_u(x,u) \leq -\alpha$$

for all $|x| \geq M$, $|u| \leq \delta$.

By a solution of (1) we mean a classical solution with bounded continuous initial data; in other words, we are considering solutions that are bounded (in $x \in \mathbb{R}^n$) for each $t \geq 0$. Let $\{U(t)\}_{t \geq 0}$ be the semigroup generated by (1); namely, for each $t \geq 0$, the operator $U(t)$ is defined by the correspondence

$$U(t) : \phi \longmapsto u(\cdot,t;\phi) \ , \tag{3}$$

where $u(x,t;\phi)$ is the solution of (1) with initial data $u_0 = \phi$.

Definition 1. A function $v \in C^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ is called an equilibrium solution of (1) if it satisfies

$$\Delta v + f(x,v) = 0 \quad \text{in} \quad \mathbb{R}^n \ . \tag{4}$$

Definition 2. A bounded continuous function $w = w(x,t)$ is called a (time-dependent) supersolution if

$$U(t)w(\cdot,t') \leq w(\cdot,t + t')$$

for any $t \geq 0$, $t' \geq 0$, where $U(t)$ is as in (3) and the relation $\phi \leq \psi$ denotes the pointwise order relation $\phi(x) \leq \psi(x)$ in $\mathbb{R}^n$. w is called a (time-dependent) subsolution if the reversed inequality holds for any $t \geq 0$, $t' \geq 0$.

Definition 3. A supersolution is called a time-independent supersolution if it is independent of the variable t; in other words, a bounded continuous function $w = w(x)$ is called a time-independent supersolution if

$$U(t)w \leq w$$

for any $t \geq 0$, where $U(t)$ is as in (3). If, in addition, w is not an equilibrium

solution, then it is called a time-independent <u>strict supersolution</u>. A time-independent (strict) <u>subsolution</u> is defined likewise.

<u>Remark 2.1</u>. $w = w(x,t)$ is a time-dependent supersolution if and only if

$$w_t \geq \Delta w + f(x,w), \qquad x \in R^n,\ t > 0$$

in a certain generalized sense. Similarly, $w = w(x)$ is a time-independent supersolution if and only if

$$\Delta w + f(x,w) \leq 0, \qquad x \in R^n$$

in a generalized sense (see Sattinger [8]).

<u>Definition 4</u>. An equilibrium solution v of (1) is said to be <u>L^∞-stable from above</u> (resp. from <u>below</u>) if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that for any $w \in C(R^n) \cap L^\infty(R^n)$ with $\|w - v\| < \delta$ and $w \geq v$ (resp. $w \leq v$) we have

$$\|U(t)w - v\| < \varepsilon$$

for all $t \geq 0$, where $\|\cdot\|$ denotes the L^∞ norm; namely

$$\|\phi\| = \sup_{x \in R^n} |\phi(x)| .$$

<u>Remark 2.2</u>. We say v is <u>L^∞-stable</u> if it is stable (with respect to L^∞ norm) in the sense of Liapounov. By the comparison theorem (Proposition 3.1), v is L^∞-stable if and only if it is L^∞-stable both from above and from below.

<u>Definition 5</u>. An equilibrium solution v is said to be <u>strongly stable from above</u> if there exists a decreasing sequence of time-independent strict supersolutions $\psi_1 > \psi_2 > \psi_3 > \cdots$ such that $\psi_m(x) \to v(x)$ as $m \to \infty$ uniformly in R^n. We say v is <u>strongly stable from below</u> if there exits an increasing sequence of time-independent strict subsolutions $\phi_1 < \phi_2 < \phi_3 < \cdots$ converging to v uniformly in R^n. If an equilibrium solution is strongly stable both from above and from below, then it is called <u>strongly stable</u>.

<u>Proposition 2.3</u>. Let (A.1), (A.2), (A.3) hold, and let v be an equilibrium solution of (1) satisfying

(5) $$\lim_{|x| \to \infty} v(x) = 0 .$$

Suppose v is strongly stable from above (resp. from below). Then it is L^∞ stable from above (resp. from below).

Remark 2.4. If we drop the assumption (A.2), (A.3) or (5), then strong stability does not necessarily imply L^∞ stability (see Appendix). This is in marked contrast to the case of diffusion problems in bounded domains, where Proposition 2.3 holds true without such assumptions as (A.2), (A.3) or (5). The proof of Proposition 2.3 will be carried out in the next section.

Remark 2.5. The concept of strong stability is related to that of structural stability in the following sense: Suppose v is a strongly stable equilibrium solution of (1). If one perturbs the equation slightly, then the perturbed problem always has a stable equilibrium solution in the vicinity of v, provided that the amplitude of perturbation is small (cf. Remark 4.3 of [5]).

Theorem. Let (A.1), (A.2), (A.3) hold, and let $\bar{w}$ be a time-independent strict supersolution and w be a time-independent strict subsolution such that $\bar{w} > w$. Assume

$$\delta > \limsup_{|x|\to\infty} \bar{w}(x) \geq \liminf_{|x|\to\infty} w(x) > -\delta \,, \tag{6}$$

where δ is as in (A.3). Then there exists an L^∞ stable equilibrium solution v of (1) satisfying $\bar{w} > v > w$ and

$$v(x) \to 0 \quad \text{as} \quad |x| \to \infty \,.$$

Remark 2.6. Since the proof of the above theorem does not make use of the Liapounov functional, this theorem remains true even if the operator Δ is replaced by any second-order uniformly elliptic linear operator that is not necessarily symmetric.

3. Proof of Theorem

Proposition 3.1 (comparison theorem). Let ϕ, ψ be bounded continuous functions on $\mathbb{R}^n$ satisfying $\phi \geqq \psi$. Then $U(t)\phi \geqq U(t)\psi$ for each $t \geqq 0$, where $U(t)$ is as in (3).

This proposition follows from the standard maximum principle and we omit the proof.

Proof of Proposition 2.3. Let ε be any positive number and put $\bar{\varepsilon} = \min\{\delta,\varepsilon\}$, where δ is as in (A.3). Since v is strongly stable from above, there exists a time-independent strict supersolution $\psi = \psi(x)$ such that $v < \psi < v + \bar{\varepsilon}/2$. Let t^* be a positive number and set $\phi = U(t^*)\psi$. By the comparison theorem, the function ϕ is again a time-independent strict supersolution; and, as is easily seen,

$$\Delta\phi + f(x,\phi) < 0 \quad \text{in} \quad \mathbb{R}^n \tag{7}$$

in the classical sense and

$$v < \phi < v + \bar{\varepsilon}/2 \quad \text{in} \quad \mathbb{R}^n \tag{8}$$

(see Remark 2.1; the strict inequality in (7) follows from the strong maximum principle for parabolic equations). In view of (5), (7), (8) and (A.3) as well as the continuity of f, we see that if $c > 0$ is a sufficiently small constant then the function $\phi_c(x) = \phi(x) + c$ satisfies

$$\Delta\phi_c + f(x,\phi_c) < 0 \quad \text{in} \quad \mathbb{R}^n , \tag{9}$$

$$v + c < \phi_c < v + \bar{\varepsilon} \quad \text{in} \quad \mathbb{R}^n . \tag{10}$$

The inequality (9) implies that ϕ_c is a time-independent supersolution. Take any $w \in C(\mathbb{R}^n)$ with $v \leqq w \leqq v + c$. By (9), (10) and the comparison theorem,

$$v \leqq U(t)w \leqq U(t)\phi_c \leqq \phi_c$$

for all $t \geqq 0$, and therefore

$$\|U(t)w - v\| < \varepsilon$$

for all $t \geqq 0$. The observations above show that v is L^∞ stable from above, completing the proof of Proposition 2.3.

Lemma 3.2. Let $v_1 < v_2$ be distinct equilibrium solutions of (1) such that there exists no equilibrium solution v satisfying $v_1 < v < v_2$. Then there exists either a

time-independent strict supersolution or a time-independent strict subsolution between v_1 and v_2.

Proof. Let $g = g(x,u)$ be a nonnegative smooth function defined on $\mathbf{R}^n \times \mathbf{R}$ such that its support $\mathrm{supp}(g)$ is compact and satisfies

(11) $$\mathrm{supp}(g) \cap \{(x, v_2(x)) \mid x \in \mathbf{R}^n\} = \emptyset$$

(12) $$g(x, v_1(x)) > 0 \quad \text{for some} \quad x \in \mathbf{R}^n .$$

Consider the Cauchy problem

(13) $$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u + f(x,u) + g(x,u), & x \in \mathbf{R}^n,\ t > 0 , \\ u(x,0) = u_0(x), & x \in \mathbf{R}^n . \end{cases}$$

By the conditions (11), (12), v_2 is an equilibrium solution of (13), whereas v_1 i time-independent strict subsolution. Let u be the solution of (13) with initial d $u_0 = v_1$. A standard argument shows that $u(x,t)$ is strictly monotone increasing in and converges as $t \to +\infty$ to an equilibrium solution, say $\bar{v}$; and $u(\cdot,t)$ is a time-independent strict subsolution of (13) for each $t \geq 0$ (see Sattinger [8]). The convergence here is locally uniform on $\mathbf{R}^n$, and clearly we have $v_1 < \bar{v} \leq v_2$. If $\bar{v} \neq v_2$, then $\bar{v}$ is not an equilibrium solution of (1) and therefore it is a time-independent strict supersolution of (1) since $g \geq 0$. On the other hand, if $\bar{v} = v_2$, then it follows from the compactness of $\mathrm{supp}(g)$ that

$$\mathrm{supp}(g) \cap \{(x, u(x,t)) \mid x \in \mathbf{R}^n\} = \emptyset$$

for all large t. This implies that for each large t the function $u(x,t)$ is a time-independent strict subsolution of (1). In either case, we have either strict super- or subsolution between v_1 and v_2. This completes the proof of Lemma 3.2.

Lemma 3.3. Let Ω be a domain in $\mathbf{R}^n$ with smooth boundary $\partial\Omega$ such that $\Omega^* \equiv \mathbf{R}^n \setminus \bar{\Omega}$ is connected. Let $u = u(x,t)$ be a continuous function defined on $\mathbf{R}^n \times \mathbf{R}^+$ such that

(a) u is C^2 in x and C^1 in t in each domain $\Omega \times \mathbf{R}^+$, $\Omega^* \times \mathbf{R}^+$; moreover u is C^1 in x on each $\bar{\Omega} \times \mathbf{R}^+$, $\overline{\Omega^*} \times \mathbf{R}^+$;

(b) u satisfies the equation (1) in each of the domains $\Omega \times \mathbf{R}^+$, $\Omega^* \times \mathbf{R}^+$;

(c) $\frac{\partial u_1}{\partial n_1} + \frac{\partial u_2}{\partial n_2} \leq 0$ on $\partial\Omega \times R^+$, where u_1, u_2 are the restrictions of u onto $\bar{\Omega}$, $\overline{\Omega^*}$ respectively and $\partial/\partial n_1$, $\partial/\partial n_2$ are the normal derivatives on $\partial\Omega$ toward Ω, Ω^* respectively.

Then u is a time-dependent supersolution of (1). In other words, we have

$$U(t)u(\cdot,t') \leq u(\cdot,t + t')$$

for any $t \geq 0$, $t' \geq 0$ (see Definition 2).

Proof. Fix $t' \geq 0$ and set $w_\varepsilon(x) = u(x,t') - \varepsilon$ for each $\varepsilon > 0$. We first show

$$U(t)w_\varepsilon \leq u(\cdot,t + t') \quad \text{for} \quad t \geq 0 . \tag{14}$$

Write $\tilde{u}_\varepsilon(\cdot,t) = U(t)w_\varepsilon$. We shall prove even a stronger version of (14):

$$\tilde{u}_\varepsilon(x,t) < u(x,t + t') \quad \text{for} \quad x \in R^n, \quad t \geq 0 . \tag{14'}$$

Clearly (14)' is satisfied at $t = 0$. Suppose (14)' holds true for $t \in [0,t_0)$, but fails to hold at $t = t_0$. Then we have

$$\tilde{u}_\varepsilon(x_0,t_0) = u(x_0,t_0 + t')$$

for some $x_0 \in R^n$. By the strong maximum principle, x_0 can neither be in Ω nor in Ω^*, for otherwise we should have

$$\tilde{u}_\varepsilon(x,t) = u(x,t + t')$$

everywhere in $\Omega \times [0,t_0]$ or everywhere in $\Omega^* \times [0,t_0]$, neither of which is possible. On the other hand, if $x_0 \in \partial\Omega$, then it follows from Hopf boundary lemma (see, for instance, Protter and Weinberger [7]) that

$$\begin{aligned} &\frac{\partial u_1}{\partial n_1}(x_0,t_0 + t') > \frac{\partial u_{\varepsilon 1}}{\partial n_1}(x_0,t_0) , \\ &\frac{\partial u_2}{\partial n_2}(x_0,t_0 + t') > \frac{\partial u_{\varepsilon 2}}{\partial n_2}(x_0,t_0) , \end{aligned} \tag{15}$$

where $u_{\varepsilon 1}$, $u_{\varepsilon 2}$ are the restrictions of u_ε onto $\bar{\Omega}$, $\overline{\Omega^*}$ respectively. Since u_ε is a smooth function, we have $\partial u_{\varepsilon 1}/\partial n_1 = -\partial u_{\varepsilon 2}/\partial n_2$. Combining this and (15), we get

$$\frac{\partial u_1}{\partial n_1}(x_0,t_0+t') + \frac{\partial u_2}{\partial n_2}(x_0,t_0+t') > 0 ,$$

which contradicts the assumption (c). This contradiction proves (14)'. Letting $\varepsilon \to 0$, we get to the conclusion of Lemma 3.3.

<u>Lemma 3.4</u>. Let (A.1), (A.2), (A.3) hold, and let v be an equilibrium solution of (1) satisfying

$$\limsup_{|x|\to\infty} |v(x)| < \delta , \tag{16}$$

where δ is the constant in (A.3). Then

$$\lim_{|x|\to\infty} v(x) = 0 . \tag{17}$$

More precisely, there exists a constant $A > 0$ such that

$$|v(x)| \leq A|x|^{-\frac{n-1}{2}} e^{-\alpha|x|} \quad \text{for all} \quad x \in \mathbb{R}^n .$$

<u>Lemma 3.5</u>. Let $v = v(x)$ be as in Lemma 3.4, and take a constant R, $R \geq M$, such that $|v(x)| \leq \delta$ for all $x \in \mathbb{R}^n - B_R$, where M, δ are as in (A.3) and

$$B_R = \{x \in \mathbb{R}^n \mid |x| < R\} . \tag{18}$$

Consider the initial-boundary value problem

$$\begin{cases} \dfrac{\partial w}{\partial t} = \Delta w + f(x,w), & x \in \mathbb{R}^n - \bar{B}_R,\ t > 0 , \\ w(x,0) = w_0(x), & x \in \mathbb{R}^n - \bar{B}_R , \\ w = v, & x \in \partial B_R,\ t > 0 . \end{cases}$$

If the initial data w_0 satisfies $|w_0(x)| \leq \delta$ for all $x \in \mathbb{R}^n - B_R$, then the solution $w(x,t)$ together with its derivatives $\partial w/\partial x_i$ $(i = 1,\dots,n)$ converges to $v(x)$ uniformly in $\mathbb{R}^n - B_R$ as $t \to \infty$. Moreover, the rate of convergence is not slower than $e^{-\alpha t}$, where α is the constant in (A.3).

These lemmas can easily be proved by constructing appropriate comparison functions and applying the maximum principle; the convergence of $\partial w/\partial x_i$ follows from that of w and the standard a priori estimates. We omit the detail of the proof.

Lemma 3.6. Let (A.1), (A.2), (A.3) hold, and let v be an equilibrium solution of (1) satisfying (17). Suppose there exists a sequence of equilibrium solutions $v_1 > v_2 > v_3 > \cdots$ that converges to v uniformly in R^n. Denote by λ_R the least eigenvalue of the eigenvalue problem

$$(19) \qquad \begin{cases} \Delta\phi + f_u(x,v(x))\phi + \lambda\phi = 0, & x \in B_R , \\ \phi = 0, & x \in \partial B_R , \end{cases}$$

where B_R is as in (18). Then λ_R is monotone decreasing in R and

$$(20) \qquad \lambda_R \to 0 \quad \text{as} \quad R \to \infty .$$

Proof. The monotonicity of λ_R in R follows from a simple argument based on the maximum principle (or the variational method), so we only prove (20).

Suppose (20) does not hold. Then

$$\bar{\sigma} \equiv \lim_{R\to\infty} \lambda_R > 0 ,$$

since we have $\lambda_R > 0$ for each $R > 0$ by virtue of the existence of the sequence $v_1 > v_2 > \cdots$. Set

$$\sigma = \min(\bar{\sigma},\alpha) ,$$

where α is as in (A.3), and consider the (linear) initial value problem

$$(21) \qquad \begin{cases} w_t = \Delta w + f_u(x,v(x))w + \sigma w, & x \in R^n, \ t > 0 , \\ w(x,0) = w_0(x), & x \in R^n . \end{cases}$$

As is easily seen, $w_n = v_n - v$ satisfies

$$\Delta w_n + f_u(x,v)w_n + \sigma w_n > 0$$

if n is sufficiently large; hence, for each large n, w_n is a time-independent strict subsolution of (21). Arguing as in the proof of Proposition 2.3, we see that for each large n there exists a small constant $\varepsilon_n > 0$ such that $\psi_n(x) = w_n(x) - \varepsilon_n$ is a time-independent strict subsolution of (21). We can choose ε_n sufficiently small so that

$$\psi_n(x) > 0 \quad \text{for some} \quad x \in R^n . \tag{22}$$

Since $w_n(x) = v_n(x) - v(x) \to 0$ as $|x| \to \infty$ by virtue of Lemma 3.4, we have

$$\lim_{|x|\to\infty} \psi_n(x) = -\varepsilon_n . \tag{23}$$

Set

$$\phi_n(x) = \max\{\psi_n(x), 0\} .$$

ϕ_n is a time-independent subsolution of (21), since the pointwise maximum of a pair of subsolutions is again a subsolution (see, for instance, [5; Proposition 2.5] or the proof of Lemma 3.7 below). By (22) and (23), ϕ_n has a compact support and is not identically equal to zero. Choose a positive number R such that

$$\mathrm{supp}(\phi_n) \subset B_R .$$

Considering that ϕ_n is a time-independent subsolution of (21) and that it vanishes near ∂B_R, we easily find that the restriction of ϕ_n onto $\bar{B}_R$ (again denoted by ϕ_n) is a time-independent subsolution of the initial-boundary value problem

$$\begin{cases} w_t = \Delta w + f_u(x,v(x))w + \sigma w, & x \in B_R,\ t > 0 , \\ w(x,0) = w_0(x), & x \in B_R , \\ w = 0, & x \in \partial B_R,\ t > 0 . \end{cases} \tag{24}$$

Moreover, ϕ_n is a strict subsolution of (24) since it is not an equilibrium solution of (24) by virtue of the unique continuation theorem for elliptic equations. The existence of such a function ϕ_n implies that $w = 0$ is an unstable equilibrium solution of (24); hence the least eigenvalue λ_R of (19) should satisfy

$$\lambda_R < \sigma .$$

But this is impossible since $\sigma \leq \bar{\sigma} = \lim_{s\to\infty} \lambda_s < \lambda_R$. This contradiction shows that the supposition $\bar{\sigma} > 0$ is false, completing the proof of Lemma 3.6.

<u>Lemma 3.7</u>. Let (A.1), (A.2), (A.3) hold, and let $v_0 < v_1$ be a pair of equilibrium solutions of (1) with $\lim_{|x|\to\infty} v_0(x) = \lim_{|x|\to\infty} v_1(x) = 0$ such that there exists neither a time-independent strict supersolution nor a time-independent strict subsolution between v_0 and v_1. Denote by S the set of all the equilibrium solutions of (1) and put

$$Y = \{v \in S \mid v_0 \leq v \leq v_1\} .$$

Then Y is a totally ordered connected subset of S. Moreover, each $v \in Y - \{v_0\}$ is L^∞ stable from below and each $v \in Y - \{v_1\}$ is L^∞ stable from above.

Proof. Take any pair $v, v^* \in Y$ and set

$$w(x) = \max\{v(x), v^*(x)\} .$$

By the comparison theorem we have

$$U(t)w \geq U(t)v = v ,$$

$$U(t)w \geq U(t)v^* = v^* ,$$

hence

$$U(t)w \geq w \quad \text{for} \quad t \geq 0 .$$

This means that w is a time-independent subsolution. Since w cannot be a strict subsolution by the assumption of the lemma, it is an equilibrium solution. Therefore, by the unique continuation theorem, we have either $w = v$ or $w = v^*$; in other words, either $v \geq v^*$ or $v \leq v^*$ holds. This shows that Y is a totally ordered set.

Next we show that Y is connected in the topology of $C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Using the boundedness of the derivatives of $f(x,u)$ and the fact that v_1, v_2 vanish at $|x| = \infty$, one easily finds that Y is compact. Suppose Y is not connected. Since Y is totally ordered and compact, the disconnectedness of Y implies that there exist $v_2, v_3 \in Y$ with $v_2 < v_3$ such that there exists no other equilibrium solution between v_2 and v_3. But this is impossible by Lemma 3.2 and the assumption of the present lemma. This contradiction shows that Y is connected.

It remains to show the stability. We only prove that each $Y - \{v_1\}$ is L^∞ stable from above, for the other part follows from a similar argument. Since Y is a totally ordered compact connected set, we can express Y as

$$Y = \{v_\theta \mid 0 \leq \theta \leq 1\} ,$$

where $v_\theta < v_\rho$ whenever $0 \leq \theta < \rho \leq 1$.

Take any θ, $0 \leq \theta < 1$, and let us prove the stability of v_θ. Let ε be any positive number, and let ρ, $\theta < \rho < 1$, be such that

$$v_\theta(x) < v_\rho(x) < v_\theta(x) + \frac{\varepsilon}{2}$$

for all $x \in R^n$. Since $v_\rho(x) \to 0$ as $|x| \to \infty$, there exists a constant R with $R \geq M$ such that

$$|v_\rho(x)| < \frac{\delta}{2}$$

for all $x \in R^n$ with $|x| \geq R$, where M, δ are as in (A.3). Now consider the following initial-boundary value problems

$$(25) \qquad \begin{cases} \dfrac{\partial p}{\partial t} = \Delta p + f(x,p), & x \in B_R,\ t > 0\ , \\ p(x,0) = p_0(x), & x \in B_R\ , \\ p = v_\rho, & x \in \partial B_R,\ t > 0\ ; \end{cases}$$

$$(26) \qquad \begin{cases} \dfrac{\partial q}{\partial t} = \Delta q + f(x,q), & x \in R^n - \bar{B}_R,\ t > 0\ , \\ q(x,0) = q_0(x), & x \in R^n - \bar{B}_R\ , \\ q = v_\rho, & x \in \partial B_R,\ t > 0\ . \end{cases}$$

In both of the problems (25), (26), v_ρ (or more precisely the restriction of v_ρ onto $\bar{B}_R$ or $R^n - B_R$) is an equilibrium solution. And v_θ, $v_\rho + \frac{\varepsilon'}{2}$ are a time-independent strict subsolution of (25) and a supersolution of (26) respectively, where

$$\varepsilon' = \min\left(\frac{\varepsilon}{2}\ ,\ \frac{\delta}{2}\right)\ .$$

Let $p(x,t)$, $q(x,t)$ be solutions of (25), (26) with initial data v_θ, $v_\rho + \frac{\varepsilon'}{2}$ respectively. By Lemma 3.5 we have

$$(27) \qquad \lim_{t \to +\infty} \{q(x,t) - v_\rho(x)\} = 0 \quad (\text{uniformly in } R^n \backslash \bar{B}_R)\ .$$

Also it is not difficult to see that

$$(28) \qquad \lim_{t \to \infty} \{v_\rho(x) - p(x,t)\} = 0 \quad (\text{uniformly in } \bar{B}_R)\ .$$

Note that the positivity of the term $v_\rho - p(x,t)$ in (27) implies that this term is asymptotically proportional to $\exp(-\lambda_R t)\phi_R(x)$, where λ_R is the least eigenvalue of (19) and ϕ_R is the corresponding positive eigenfunction. And we have

$$\lambda_R \to 0 \quad \text{as} \quad R \to \infty$$

by Lemma 3.6. On the other hand, by Lemma 3.5, the convergence rate in (27) is not slower than that of $\exp(-\alpha t)$. Take a sufficiently large R such that $\lambda_R < \alpha$, and fix it. Then the convergence rate in (27) is faster than that of (28). Consequently, there exists a $T > 0$ such that

$$(29) \qquad \frac{\partial p}{\partial N} + \frac{\partial q}{\partial n} \leq 0, \qquad |x| = R, \quad t \geq T ,$$

where $\partial/\partial N$, $\partial/\partial n$ denote the inner and the outer normal derivatives to ∂B_R respectively. Now define a function w by

$$w(x,t) = \begin{cases} p(x,t + T) & (|x| \leq R, \quad t \geq 0), \\ q(x,t + T) & (|x| \geq R, \quad t \geq 0) . \end{cases}$$

By virtue of Lemma 3.3 and (29), w is a time-dependent supersolution of (1). It is clear that

$$(30) \qquad v_\theta(x) < w(x,t) < v_\theta(x) + \varepsilon$$

for all $x \in \mathbb{R}^n$ and $t \geq 0$. It is also clear that

$$(31) \qquad \inf_{x \in \mathbb{R}^n} \{w(x,0) - v_\theta(x)\} > 0 .$$

As ε is an arbitrary positive number, (30) and (31) together with the comparison theorem and the fact that w is a supersolution imply that v_θ is L^∞ stable from above. This completes the proof of Lemma 3.7.

<u>Proof of Theorem</u>. Let $\{U(t)\}_{t\geq 0}$ be the semigroup generated by (1) (see (3)). As is easily seen, $U(t)w$ is strictly monotone increasing in t and converges as $t \to +\infty$ to an equilibrium solution of (1), say $\underline{v}$ (see Sattinger [8]). Similarly, $U(t)\bar{w}$ converges to an equilibrium solution, say $\bar{v}$. Obviously we have $w < \underline{v} \leq \bar{v} < \bar{w}$ and that $\underline{v}$ is strongly stable from below while $\bar{v}$ is strongly stable from above. We also have

$$(32) \qquad \lim_{|x|\to\infty} \underline{v}(x) = \lim_{|x|\to\infty} \bar{v}(x) = 0$$

by Lemma 3.4. Denote by S the set of all the equilibrium solutions v of (1) satisfying $\underline{v} \leq v \leq \bar{v}$, and set

$$S_0 = \{v \in S \mid v \text{ is strongly stable from below}\} .$$

S_0 is not empty since it contains $\underline{v}$. Arguing as in the proof of [5; Theorem 2'], we easily find that S_0 is an inductively ordered set; in other words, any totally ordered subset of S_0 has an upper bound in S_0. By Zorn's lemma, S_0 has a maximal element, say v_0. Set

$$S_1 = \{v \in S \mid v \geq v_0, v \text{ is strongly stable from above}\} .$$

A similar argument shows that S_1 has a minimal element, say v_1.

If $v_0 = v_1$, then v_0 is strongly stable both from above and from below; hence it is L^∞ stable by Proposition 2.3.

Next consider the case $v_0 < v_1$. In this case there exists neither a time-independent strict supersolution nor a time-independent strict subsolution between v_0 and v_1. In fact, if $w = w(x)$ is a strict subsolution satisfying $v_0 < w < v_1$, then $U(t)w$ is monotone increasing in t and converges as $t \to +\infty$ to an equilibrium solution v, $v_0 < v \leq v_1$, which is strongly stable from below. But this is impossible by the maximality of v_0 in S_0. Similarly, the existence of a strict supersolution contradicts the minimality of v_1 in S_1. This contradiction shows that the above claim is true. Therefore, by Lemma 3.7,

$$Y = \{v \in S \mid v_0 \leq v \leq v_1\}$$

is a compact connected set and each element of Y is L^∞ stable. This completes the proof of Theorem.

Appendix

As mentioned in Remark 2.4, if we drop the assumption (A.2), (A.3) or (5), then Proposition 2.3 is no longer true; in other words, strong stability does not necessarily imply L^∞ stability. In this case, our main theorem also fails to hold.

To see this, consider the following example:

$$\left\{\begin{array}{ll} \dfrac{\partial u}{\partial t} = \Delta u + u^3, & x \in R^5,\ t > 0\ , \\ u(x,0) = u_0(x), & x \in R^5\ . \end{array}\right. \tag{33}$$

$u \equiv 0$ is an equilibrium solution of this problem. A simple calculation shows that

$$w_\lambda(x) = \frac{\lambda}{|x|^2 + 1}$$

is a time-independent strict supersolution of (33) for $0 < \lambda \leq \sqrt{10}$, while it is a time-independent strict subsolution for $-\sqrt{10} \leq \lambda < 0$. Moreover, we have

$$w_\lambda(x) \to 0 \quad \text{as} \quad \lambda \to 0$$

uniformly in $x \in R^5$. Therefore, $u \equiv 0$ is strongly stable. On the other hand, as is easily seen, $u \equiv 0$ is not L^∞ stable. This shows that the conclusion of Proposition 2.3 (as well as that of the Theorem) is not true if we drop the assumption (A.3).

REFERENCES

[1] Crandall, M. G., P. C. Fife and L. A. Peletier, Existence of stable equilibria for scalar reaction-diffusion equations in $\mathbb{R}^n$, in preparation.

[2] Crandall, M. G. and H. Matano, in preparation.

[3] Hirsch, M. W., Equations that are competitive or cooperative II -- Convergence almost everywhere, to appear in SIAM Journal on Math. Analysis.

[4] Matano, H., Asymptotic behavior and stability of solutions of semilinear diffusion equations, Publ. RIMS Kyoto Univ., 15 (1979), 401-454.

[5] Matano, H., Existence of nontrivial unstable sets for equilibriums of strongly order-preserving systems, J. Fac. Sci. Univ. Tokyo, 30 (1984), 645-673.

[6] Matano, H. and M. Mimura, Pattern formation in competition-diffusion systems in nonconvex domains, Publ. RIMS Kyoto Univ., 19 (1983), 1049-1079.

[7] Protter, M. H. and H. F. Weinberger, Maximum principles in differential equations, Prentice-Hall, 1967.

[8] Sattinger, D. H., Monotone methods in nonlinear elliptic and parabolic boundary value problems, Indiana Univ. Math. J., 21 (1972), 979-1000.

HM:scr

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER 2678	2. GOVT ACCESSION NO. AD-A141 601	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) L^∞ STABILITY OF AN EXPONENTIALLY DECREASING SOLUTION OF THE PROBLEM $\Delta u + f(x,u) = 0$ IN $\mathbb{R}^n$		5. TYPE OF REPORT & PERIOD COVERED Summary Report - no specific reporting period
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Hiroshi Matano		8. CONTRACT OR GRANT NUMBER(s) DAAG29-80-C-0041
9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematics Research Center, University of Wisconsin 610 Walnut Street Madison, Wisconsin 53706		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Work Unit Number 1 - Applied Analysis
11. CONTROLLING OFFICE NAME AND ADDRESS U. S. Army Research Office P.O. Box 12211 Research Triangle Park, North Carolina 27709		12. REPORT DATE April 1984
		13. NUMBER OF PAGES 18
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Nonlinear parabolic equation
stable solution
super and subsolution
comparison principle

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

We consider the Cauchy problem $u_t = \Delta u + f(x,u)$, $x \in \mathbb{R}^n$, $t > 0$, and prove that if there exist a strict supersolution $\bar{w}$ and a strict subsolution $\underline{w}$ with $\bar{w} > \underline{w}$ then there exists at least one stable equilibrium solution between $\bar{w}$ and $\underline{w}$ provided that f satisfies certain conditions. The stability is with respect to the L^∞ norm. Unlike the case where the spatial domain is bounded, some difficulties occur near $|x| = \infty$ in the present problem. The major part of this paper is devoted to dealing with such difficulties.

DD FORM 1473, 1 JAN 73 EDITION OF 1 NOV 65 IS OBSOLETE

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

END

FILMED